THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4060 Complex Analysis 2022-23
Tutorial 11
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1. (Ex.1 Ch.9 in textbook) Suppose that a meromorphic function f has two periods ω_1 and ω_2, with $\omega_2/\omega_1 \in \mathbb{R}$

   (a) Suppose ω_2/ω_1 is rational, say equal to p/q, where p and q are relatively prime integers. Prove that as a result the periodicity assumption is equivalent to the assumption that f is periodic with the simple period $\omega_0 = \frac{1}{q}\omega_1$. [Hint: Since p and q are relatively prime, there exist integers m and n such that $mq + np = 1$ (Corollary 1.3, Chapter 8, Book I).]

   (b) If ω_2/ω_1 is irrational, then f is constant. To prove this, use the fact that $\{m - n\tau\}$ is dense in $\mathbb{R}$ whenever τ is irrational and m, n range over the integers.

   Solution. (a) If ω_2/ω_1 is rational, say, $\frac{p}{q}$ with p, q relatively prime, then there exist intergers m and n, such that $mp + nq = 1$. Then we have

   $$f(z) = f(z + mw_2 + nw_1) = f(z + \frac{mp}{q}w_1 + nw_1) = f(z + \frac{1}{q}w_1)$$

   (b) If $\tau = \omega_2/\omega_1$ is irrational, then

   $$f(z) = f(z + mw_2 + nw_1) = f(z + (m\tau + n)w_1)$$

   Since $(m\tau + n)$ is dense in $\mathbb{C}$, for fixed z, we find $f(z)$ is constant on a dense subset of $\mathbb{C}$. On the other hand, f is meromorphic, hence constant.

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2. (Ex.3 Ch.9 in textbook) In contrast with the result in Lemma 1.5, prove that the series

   $$\sum_{n+m\tau\in\Lambda^*} \frac{1}{|n+m\tau|^2} \quad \text{where } \tau \in \mathbb{H}$$

   does not converge. In fact, show that

   $$\sum_{1\leq n^2+m^2\leq R^2} 1/\left(n^2+m^2\right) = 2\pi \log R + O(1) \quad \text{as } R \to \infty.$$

   Solution. Recall in our proof of Lemma 1.5 we showed there exists a small δ_τ, such that

   $$\delta_\tau |n + mi| \leq |n + m\tau|$$

   By the similar way, we can show exists ϵ_τ, such that

   $$\epsilon_\tau |n + m\tau| \leq |n + mi| = m^2 + n^2$$

Thus we only need to show

$$\sum_{1\le n^2+m^2\le R^2} \frac{1}{n^2+m^2}$$

diverges. For any integer N, consider the region: $A_n := \{(n,m) \in \Lambda^* | N - \frac{1}{2} < |m|, |n| < N + \frac{1}{2}\}$. It is easy to count there are $8N$ lattice points inside A_n and

$$\sum_{1\le n^2+m^2\le R^2} \frac{1}{n^2+m^2} = \sum_{i=1}^{\infty} \sum_{(n,m)\in\Lambda^*\cap A_i} \frac{1}{n^2+m^2}$$

Thus,

$$\sum_{i=1}^{\infty}(8i)\frac{1}{2i^2} \le \sum_{i=1}^{\infty} \sum_{(n,m)\in\Lambda^*\cap A_i} \frac{1}{n^2+m^2} \le \sum_{i=1}^{\infty}(8i)\frac{1}{i^2}$$

which diverges. ◀

3. (Ex.4 Ch.9 in textbook)By rearranging the series

$$\frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right]$$

show directly, without differentiation, that $\wp(z+\omega) = \wp(z)$ whenever $\omega \in \Lambda$. [Hint: For R sufficiently large, note that $\wp(z) = \wp^R(z) + O(1/R)$, where $\wp^R(z) = z^{-2} + \sum_{0<|\omega|<R}\left((z+\omega)^{-2} - \omega^{-2}\right)$. Next, observe that both $\wp^R(z+1) - \wp^R(z)$ and $\wp^R(z+\tau) - \wp^R(z)$ are $O\left(\sum_{R-c<|\omega|<R+c} |\omega|^{-2}\right) = O(1/R)$.]

Solution. We follow the hint, dividing it into two parts: $\forall |z| \le \sqrt{R}$

$$\frac{1}{z^2}+\sum_{\omega\in\Lambda^*}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right] = A_R(z)+B_R(z) = z^{-2}+\sum_{0<|\omega|<R}\left((z+\omega)^{-2} - \omega^{-2})\right)+\sum_{R\le|\omega|}\left((z+\omega)^{-2} - \omega^{-}\right.$$

First we have

$$\left|\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right| = \left|\frac{-z^2-2z\omega}{\omega^2(z+\omega)^2}\right| \le C\frac{1}{|\omega|^3}$$

Thus

$$|B_R(z)| \le \sum_{i=R}^{\infty} \sum_{i-1<|\omega|<i+1} \frac{C}{|\omega|^3} \sim \frac{1}{R}$$

Here we use a fact that the number of integer points inside the annulus $\{i-1 \le |y| \le i+1\}$ is almost ki. And for $A_R(z)$, the difference between $A_R(z)$ and $A_R(z+1)$ is at most the values in the Annulus:

$$|A_R(z) - A_R(z+t)| \le \sum_{R-t\le|\omega|\le R+t} \frac{1}{|(z+\omega)^2|} \sim \frac{1}{R}$$

Thus $\wp(z) - \wp(z+1) \sim \frac{1}{R}$ and let $R \to \infty$, we get the result we desired. ◀